Weaver SSB Modulation/Demodulation - A Tutorial

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February 18, 2017

1 Introduction

In 1956 D. K. Weaver [1] proposed a new modulation scheme for single-sideband-suppressed-carrier (SSB) generation. The Weaver Method (also known as the Third Method), has potential advantages compared to the two prior methods, the filter and phase-shift methods, but is more difficult to understand. With the advent of DSP (Digital Signal Processing) and SDR (Software-Defined Radio) the Weaver Method has received much greater attention but is still not generally well understood by hobbyists. This tutorial is designed to introduce the Weaver Method using simple mathematics that is limited to a few trigonometric identities.

The descriptions described here are based on *real* (as opposed to *complex*) signal representations. In particular all signals are represented in terms of sinusoidal components of the form $A\cos(2\pi Ft + \phi)$ where A is the amplitude, F is the frequency with units of Hz, and ϕ is the phase (with units of radians) $(0 \le \phi < 2\pi)$. For convenience we will make the substitution $\omega = 2\pi F$ and express the frequency as an *angular* frequency, with units radians/second.

Fourier theory tells us that any practical time-based waveform, $f(t)$, may be expressed mathematically as a *sum* (finite or infinite) of such sinusoids,

$$f(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n), \tag{1}$$

and can be entirely described by its *spectrum*, that is the collection of all amplitudes A_n, frequencies ω_n, and phases ϕ_n [2]. We are all familiar with this concept in terms of filtering a signal, where we modify that signal by manipulating its spectrum, that is modifying the values of A_n and ϕ_n. For the purposes of this tutorial we need to be able to think of a signal in terms of its spectral components.

[1] Weaver, D.K., *A third method of generation and detection of single sideband signals*, Proc. IRE, Dec. 1956, pp. 1703-1705

[2] There are several ways of expressing the spectrum of a signal $f(t)$. In this tutorial we adopt the convention of a one-sided, purely real spectrum as described above. Another more general and compact description uses a collection of *complex exponentials* of the form $A_n e^{\pm j\omega_n t}$ to describe the spectrum, and requires positive and negative frequency components. Modern signal analysis relies almost exclusively on the complex notation.

1.1 Introduction to SSB Communication

Single-sideband suppressed carrier (SSB) modulation is a variant of *amplitude modulation* (AM) and is used extensively in high-frequency radio voice communication, such as maritime and ham radio voice communication.

Amplitude (AM) Modulation: In AM radio, the audio information is impressed upon a sinusoidal *carrier* signal $\cos(\omega_c t)$ with angular frequency ω_c, and amplitude A by modulating the *amplitude* of the carrier $A\cos(\omega_c t)$, with an audio signal $f_{\text{audio}}(t)$ as follows:

$$f_{\text{AM}}(t) = (1 + \alpha f_{\text{audio}}(t)) \times \cos(\omega_c t) \tag{2}$$

where α is known as the modulation index, and must be chosen so that $\alpha f_{\text{audio}}(t)) > 0$ (Otherwise distortion caused by *over modulation* occurs.) . Notice if the audio is silent that is $f_{\text{audio}}(t) = 0$, the transmitted signal is simply the carrier $f_{\text{AM}}(t) = A\cos(\omega_c t)$.

For simplicity we start by assuming we have a very simple audio signal, a simple audio tone with a frequency of F_a Hz, so that

$$f_{audio}(t) = A_a \cos(2\pi F_a t + \phi_a) = A_a \cos(\omega_a t + \phi_a) \tag{3}$$

where A_a is the amplitude, $\omega_a = 2\pi F_a$ is the *angular frequency* in units of radians/second, and ϕ_a is an arbitrary phase angle[3] When Eq(3) is substituted into Eq(2)

$$f_{\text{AM}}(t) = \cos(\omega_c t) + \alpha A_a \left(\cos(\omega_a t + \phi_a) \times \cos(\omega_c t)\right) \tag{4}$$

where there is a multiplication of two time-based sinusoidal functions. Using Eq. (A.1) in the appendix to expand the product of two cosine functions)

$$f_{\text{AM}}(t) = \cos(\omega_c t) + \frac{A_a \alpha}{2} \cos((\omega_c + \omega_a)t + \phi_a) + \frac{A_a \alpha}{2} \cos((\omega_c - \omega_a)t - \phi_a) \tag{5}$$

which shows three separate sinusoidal components 1) a carrier component at frequency ω_2, 2) an *upper sideband* component at frequency $(\omega_c + \omega_a)$, and 3) a *lower sideband* component at frequency $(\omega_c - \omega_a)$, as shown in Fig. 1.

As noted above, we can represent *any* audio signal, for example speech or music, as a sum of N such components and write

$$f_{\text{audio}}(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n),$$

and substitute into Eq(2). Then applying the above argument to each of the N components

$$f_{\text{AM}}(t) = \cos(\omega_c t) + \frac{\alpha}{2}\left(\sum_{n=1}^{N} A_n \cos((\omega_c + \omega_n)t + \phi_n)\right) + \frac{\alpha}{2}\left(\sum_{n=1}^{N} A_n \cos((\omega_c - \omega_n)t - \phi_n)\right) \tag{6}$$

where the upper and lower sidebands each contain many components, as shown in Fig. 2.

[3]We will use angular frequency throughout this document because it generates a more compact representation.

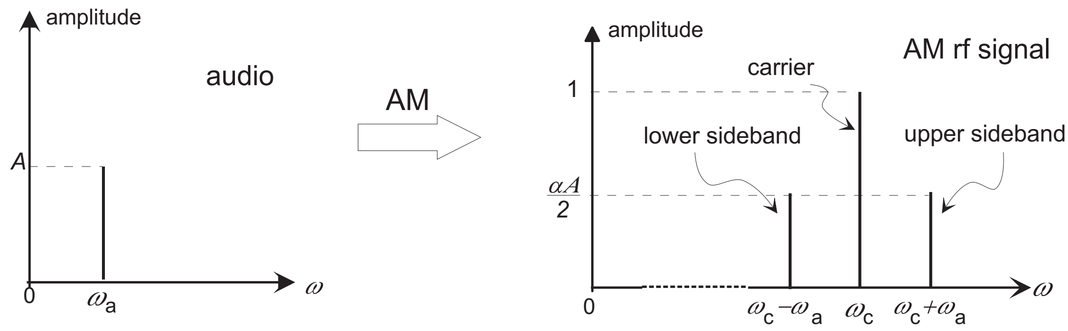

Figure 1: Amplitude spectrum of a simple AM rf signal with carrier frequency ω_c and audio frequency ω_a.

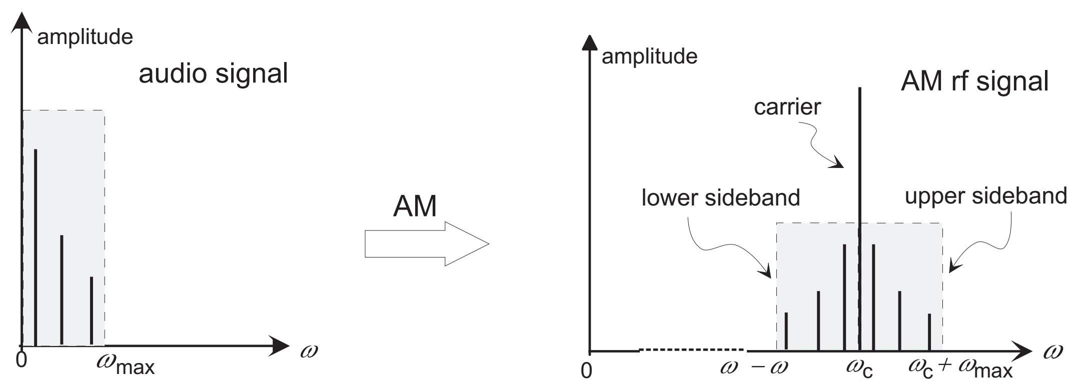

Figure 2: Amplitude spectrum of an AM rf signal with carrier frequency ω_c modulated by an audio signal containing a set of three components with frequencies $0 < \omega < \omega_{max}$. The envelope of the audio spectrum is shown as shaded, and asymmetric to help visualize the spectral relationships between the sidebands.

We are all familiar with AM modulation though listening on the AM broadcast band. Historically, AM was the first radio voice/music communication method, and while easy to implement in hardware it, has a number of disadvantages compared to other modulation techniques. In particular, for high frequency communication

- Most of the transmitted signal power is contained within the carrier (which is transmitted continuously, even in the absence of an audio signal), making AM a very energy inefficient communication mode.
- The spectral width of the AM waveform is twice the audio bandwidth, and is therefore an inefficient use of the high-frequency radio spectrum.
- In long-distance communication, with multipath reflections from the ionosphere, cancelation of the carrier can occur leading to "selective fading", and loss of intelligibility.

Single-Sideband Suppressed Carrier (SSB) Modulation: SSB generation simply involves elimination of the carrier and one of the two sidebands from an AM transmission, leaving only the upper (USB) or lower (LSB) sideband. There are therefore two possible modes of SSB transmission, either LSB or USB, as shown in Fig. 3.

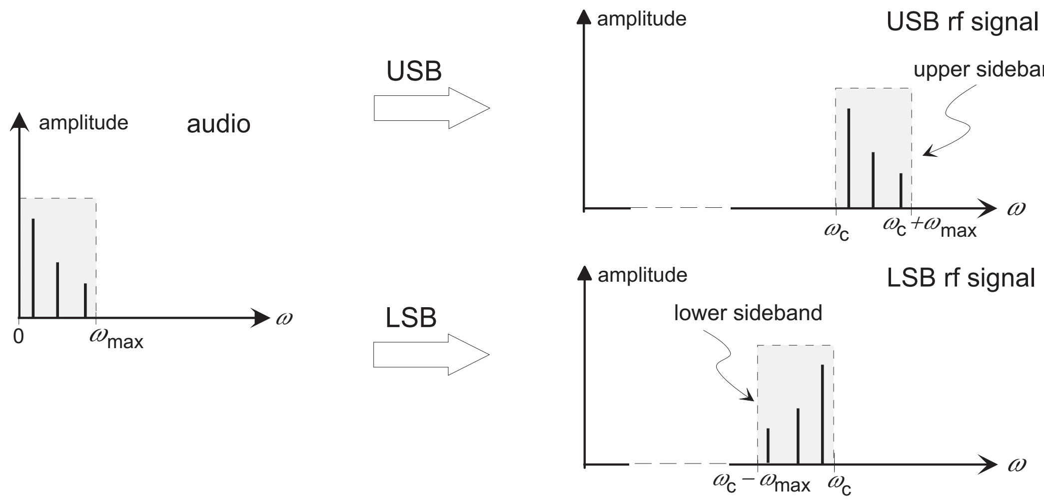

Figure 3: Spectral representation of upper-side-band (USB), and lower-sideband (LSB) signals of a three component audio signal with audio bandwidth ω_{max} at the same rf frequency ω_c .

Notice that SSB transmission directly addresses the first two criticisms of AM transmission noted above:

- There is no power transmitted as a carrier, and the power transmitted is zero in the absence of any audio signal.
- The spectral bandwidth is half that of an AM signal,

In practice SSB is also far less susceptible to the "selective fading" that plagues AM transmission at high frequencies.

We note that USB modulation is a straight frequency translation of each component in $f_{audio}(t)$:

$$f_{audio}(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n) \longrightarrow \text{USB} \longrightarrow \sum_{n=1}^{N} A_n \cos((\omega_c + \omega_n)t + \phi_n), \tag{7}$$

and LSB is a translation and reflection (flipping) of the components about the suppressed-carrier frequency ω_c:

$$f_{audio}(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n) \longrightarrow \text{LSB} \longrightarrow \sum_{n=1}^{N} A_n \cos((\omega_c - \omega_n)t - \phi_n). \tag{8}$$

1.2 Historical methods of SSB Waveform Generation:

Historically there were two methods of generating an SSB signals:

- The "**filtering**" method, is a brute-force method, in which a multiplier is used to create a *double sideband* (DSB) waveform at a fixed frequency ω_c. If $f_{audio}(t) = A_a \cos(\omega_a t + \phi_a)$

$$\begin{aligned} f_{\text{DSB}}(t) &= A_a \cos(\omega_a t + \phi_a) \times \cos(\omega_c t) \\ &= \frac{A_a}{2} \cos((\omega_c + \omega_a)t + \phi_a) + \frac{A_a}{2} \cos((\omega_c - \omega_a)t - \phi_a) \end{aligned}$$

  from Eq. (A.1). $f_{\text{DSB}}(t)$ is then passed through a very high quality band-pass filter (in hardware usually a mechanical or xtal filter) centered on the desired sideband to reject the unwanted sideband.

- The "**phase-shift**" or "**phasing**" method generates the SSB signal directly from the trigonometric relationships in Eq. (A.5) in the Appendix

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b) \end{aligned}$$

  with the constant angles a, b replaced with time varying functions of the form ωt.

  If the audio signal is a simple tone $f_{audio}(t) = A_a \cos(\omega_a t + \phi_a)$, a USB signal may be created from the expansion

$$y_{USB}(t) = A_a \cos((\omega_c + \omega_a)t + \phi_a) = A_a \cos(\omega_a t + \phi_a)\cos(\omega_c t) - A_a \sin(\omega_a t + \phi_a)\sin(\omega_c t).$$

  There is a problem because we do not have the signal $A_a \sin(\omega_a t + \phi_a)$ available to us, and in practice it cannot be generated directly from the audio $A_a \cos(\omega_a t + \phi_a)$. However, using Eq. (A.10) in the Appendix ($\sin(a) = \cos(a - \pi/2)$) we can write

$$y_{USB}(t) = A_a \cos(\omega_a t + \phi_a) \times \cos(\omega_c t) - A_a \cos(\omega_a t + \phi_a - \pi/2) \times \sin(\omega_c t). \qquad (9)$$

  and use a (hardware or software) all-pass "$\pi/2$ phase-shifter" (also known as a Hilbert transformer) to approximate the $A_a \sin(\omega_a t + \phi_a)$ as indicated in Fig. (4).

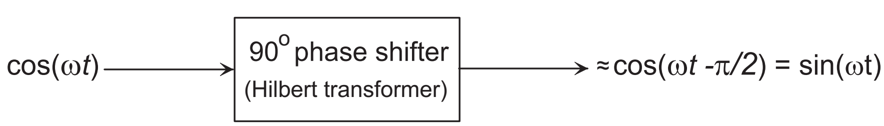

Figure 4: The Hilbert transformer for approximating a wideband $\pi/2$ phase shift.

  Similarly for lower-sideband generation [4]

$$\begin{aligned} y_{LSB}(t) &= A\cos((\omega_c - \omega_a)t + \phi) \\ &= A\cos(\omega_a t + \phi)\cos(\omega_c t) + A\sin(\omega_a t + \pi/2)\sin(\omega_c t) \\ &= A\cos(\omega_a t + \phi) \times \cos(\omega_c t) + A\cos(\omega_a t + \phi - \pi/2) \times \sin(\omega_c t) \qquad (10) \end{aligned}$$

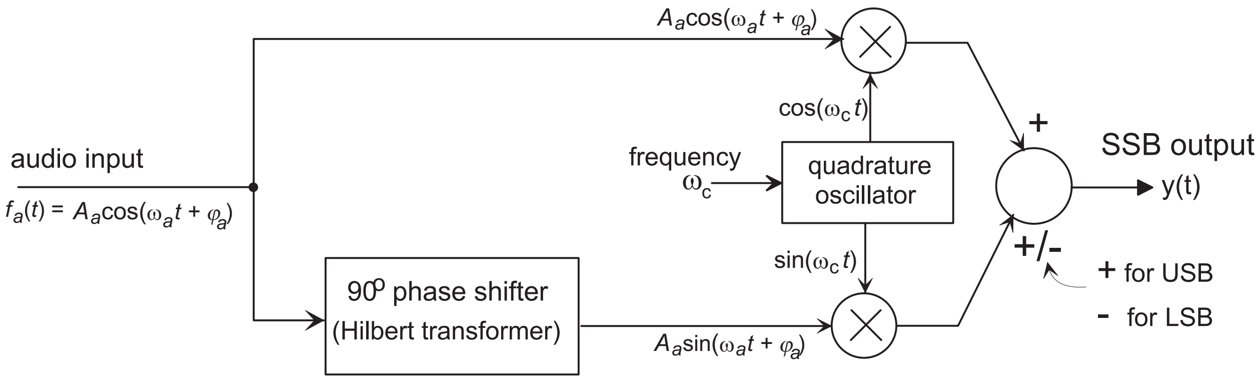

Figure 5: The signal processing steps involved in the phasing method of SSB generation.Note that the only difference between USB and LSB generation is the sign of the summation at the output.

The signal processing flow is indicated in Fig. 5.

The phase-shift method is probably the most widely used SSB generator, and is well suited to both hardware and software (DSP) methods. Its major downside is that the phase shifter must be an all-pass filter, with a constant phase shift of $\pi/2$ radians (90°) across the whole audio bandwidth. Although modern filter design allows for accurate results, this can at best be only approximated. Errors in the phase shift result in incomplete suppression of the unwanted sideband.

2 The Weaver SSB Modulator

In the 1950's the design and implementation of accurate phase- shift networks was a difficult task. The Weaver method eliminates the need for such networks, and replaces them with a pair of identical low-pass filters with modest design specifications.

2.1 The Weaver Modulator

As discussed above, our goal is to translate an audio signal $f_{audio}(t)$ containing N spectral components

$$f_{audio}(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n))$$

to a SSB waveform

$$y_{ssb} = \sum_{n=1}^{N} A_i \cos((\omega_c \pm \omega_n)t \pm \phi_n)$$

where the "+" indicates the USB, and the "−" the LSB modes.

[4]The explicit use of the multiplication symbol × in Eqs. (6) and (7) has no mathematical significance, and is only used to emphasize the signal processing operations implied.

Assume that the audio signal is band limited between frequencies ω_{min} and ω_{max} as shown in Fig. 6. For example, in a communications radio the audio might be limited to $300 - 3000$ Hz ($1,885 - 18,849$ rad/s). Then define the mid-point of the audio spectrum as

$$\omega_o = (\omega_{min} + \omega_{max})/2. \tag{11}$$

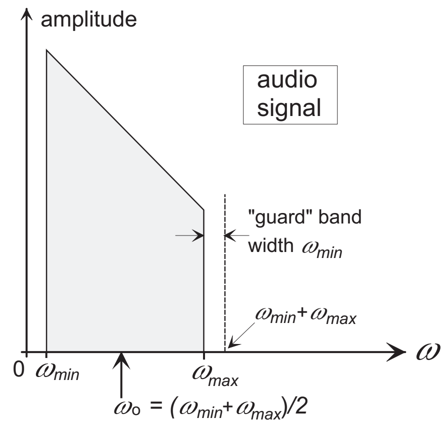

Figure 6: Schematic representation of the amplitude spectrum of the band-limited audio signal $f_{audio}(t)$. The shaded area shows the "envelope" of the A_n. Also shown is the definition of the frequency $\omega_o = (\omega_{min} + \omega_{max})/2$.

The block diagram of the Weaver modulator is shown in Fig. 7. Notice that there are a pair of parallel chains, each containing two oscillators, a low-pass filter, and two multipliers. As before, for simplicity we will consider the processing of a single representative component of $f_{audio}(t)$,

$$f_a(t) = A_a \cos(\omega_a t + \phi_a))$$

with amplitude A_a, frequency ω_a, and phase ϕ_a through the modulation process.

- The first oscillator in each chain is a *fixed, low frequency* oscillator set to the audio mid-point frequency ω_o.
- The second is a high frequency (rf) oscillator at a frequency ω_s. In practice this is used for tuning the SSB output to the desired frequency ω_c. (As we will see below $\omega_s \neq \omega_c$).

The two low-pass filters are identical, and have a cut-off frequency of ω_o. For now they are assumed to be *ideal* - that is they pass all components with a frequency $-\omega_o \leq \omega \leq \omega_o$ unimpeded, while rejecting all frequencies.

We now examine the flow of $f_a(t)$ through the modulator:

1. **At the output of the first multipliers ($\mathbf{f_1(t)}$ and $\mathbf{f_2(t)}$):** The output of the first pair of multipliers are the products of the audio signal $f_a(t) = A_a \cos(\omega_a t + \phi_a)$ and the pair of quadrature sinusoids $\cos(\omega_o t)$ and $\sin(\omega_o t)$:

$$\begin{aligned} f_1(t) &= f_a(t) \times \cos(\omega_o t) = A_a \cos(\omega t + \phi_a) \cos(\omega_o t) \\ &= \frac{A_a}{2} \cos((\omega_a - \omega_o) t + \phi_a) + \frac{A_a}{2} \cos((\omega_a + \omega_o) t + \phi_a) \end{aligned} \tag{12}$$

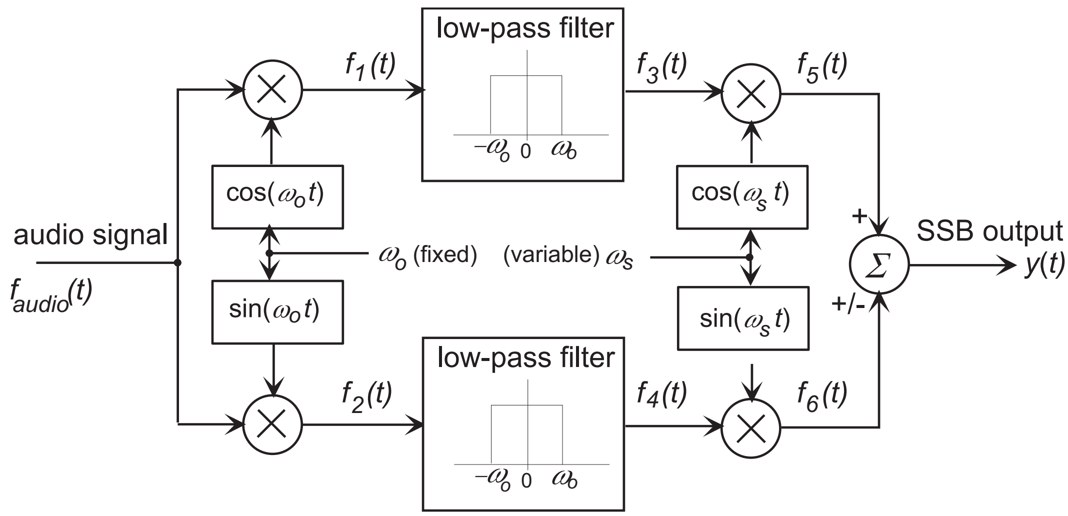

Figure 7: The signal processing flow for the Weaver SSB modulator

from Eq. (A.1), and

$$\begin{aligned} f_2(t) &= f_a(t) \times \sin(\omega_o t) = A_a \cos(\omega_a t + \phi_a) \sin(\omega_o t) \\ &= -\frac{A_a}{2} \sin((\omega_a - \omega_o) t + \phi_a) + \frac{A_a}{2} \sin((\omega_a + \omega_o) t + \phi_a) \end{aligned} \tag{13}$$

from Eq. (A.6). The amplitude spectra of $f_1(t)$ and $f_2(t)$ are shown in Fig. 8.

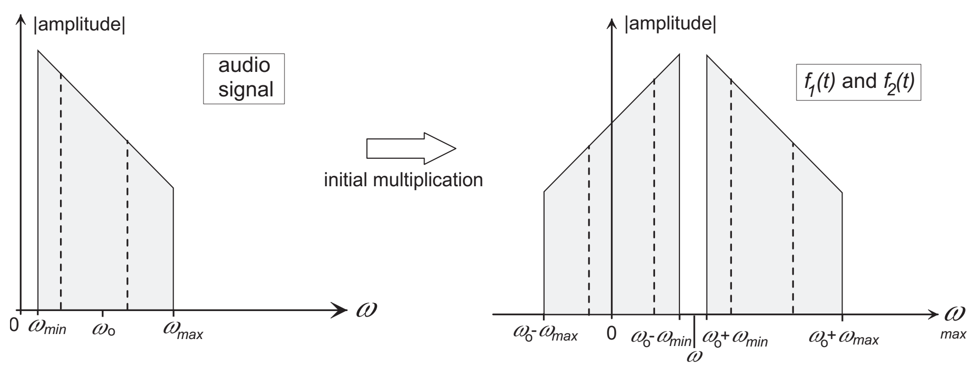

Figure 8: Schematic amplitude spectra at the output of the first multiplier stage in the Weaver modulator. Two sample audio components are shown as dashed lines. Note that although $f_1(t)$ and $f_2(t)$ have identical amplitude spectra, their phase spectra differ.

2. **At the outputs of the Low-Pass Filters ($\mathbf{f_3(t)}$ and $\mathbf{f_4(t)}$):** We assume that the two low-pass filters are "ideal", that is they pass all spectral components(positive or negative) within their bandwidth $-\omega_o \leq \omega \leq \omega_o$ unimpeded, while completely rejecting all components with $\omega < -\omega_o$ and $\omega > \omega_o$, where the response to negative frequency

components is as defined in Appendix B.[5]

The outputs from the two low-pass filters, ($f_3(t)$ and $f_4(t)$), are therefore simply the low-frequency (difference) components in $f_1(t)$ and $f_2(t)$ with frequencies below ω_o:

$$f_3(t) = \begin{cases} \frac{A_a}{2}\cos\left((\omega_a - \omega_o)\,t + \phi_a\right) & \text{if } -\omega_o \le \omega_a \le \omega_o \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

and

$$f_4(t) = \begin{cases} -\frac{A_a}{2}\sin\left((\omega_a - \omega_o)\,t + \phi_a\right) & \text{if } -\omega_o \le \omega_a \le \omega_o \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The amplitude spectra of $f_3(t)$ and $f_4(t)$ are shown in Fig. 9.

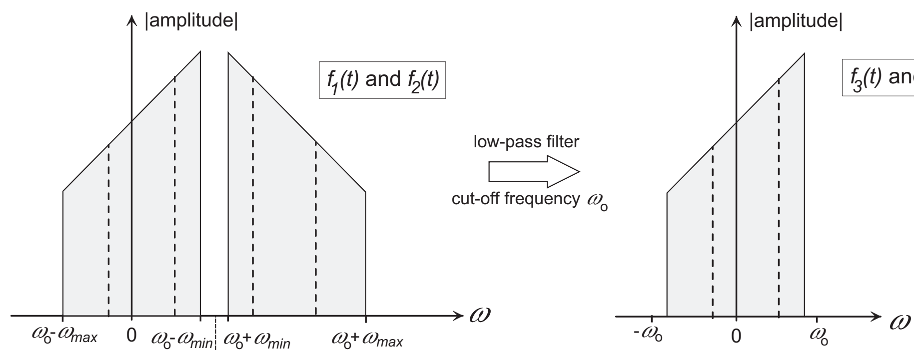

Figure 9: Low-pass filtering to retain only those components with a frequency in the range $-\omega_o \le \omega_a \le \omega_o$.

3. **At the outputs of the second multipliers ($\mathbf{f_5(t)}$ and $\mathbf{f_6(t)}$):** In general the second oscillator frequency $\omega_s >> \omega_o$. Then $f_5(t)$ and $f_6(t)$ are simply:

$$\begin{aligned} f_5(t) &= f_3(t) \times \cos(\omega_s t) \\ &= \frac{A_a}{4}\cos\left(((\omega_s + \omega_o) - \omega_a)\,t - \phi_a\right) + \frac{A_a}{4}\cos\left(((\omega_s - \omega_o) + \omega_a)\,t + \phi_a\right) \quad (16) \\ f_6(t) &= f_4(t) \times \sin(\omega_s t) \\ &= -\frac{A_a}{4}\cos\left(((\omega_s + \omega_o) - \omega_a)\,t - \phi_a\right) + \frac{A_a}{4}\cos\left(((\omega_s - \omega_o) + \omega_a)\,t + \phi_a\right) \quad (17) \end{aligned}$$

4. **Summation/Subtraction to Generate the SSB Output:** Upper or lower sideband signals are generated by simple addition or subtraction of $f_5(t)$ and $f_6(t)$:

$$\begin{aligned} y_{usb}(t) &= f_5(t) + f_6(t) = \frac{A_a}{2}\cos\left(((\omega_s - \omega_o) + \omega_a)\,t + \phi_a\right) \\ &= \frac{A_a}{2}\cos\left((\omega_c + \omega_a)\,t + \phi_a\right) \quad (18) \end{aligned}$$

[5] For theoretical reasons it is impossible to design or build an "ideal" filter. We simply assume its existence to illustrate the the signal processing involved and accept that any practical filter will not meet the idealized specifications.

where $\omega_c = \omega_s - \omega_o$, which is the desired signal frequency. similarly

$$\begin{aligned} y_{lsb}(t) &= f_5(t) - f_6(t) = \frac{A_a}{4} \cos\left(\left((\omega_s + \omega_o) - \omega_a\right) t - \phi_a\right) \\ &= = \frac{A}{2} \cos\left((\omega_c - \omega_a)\, t - \phi_a\right) \end{aligned} \tag{19}$$

where in this case $\omega_c = (\omega_s + \omega_o)$ is the output LSB frequency.

2.2 Design Summary:

Given a band-limited audio signal

$$f_{audio}(t) = \sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n)$$

where $\omega_{min} \leq \omega_n < \omega_{max}$ for all n, proceed as follows:

1. Select $\omega_o = (\omega_{min} + \omega_{max}/2$.
2. Design and implement a pair of low-pass filters with cut-off frequency ω_o, (and with a transition band of width $2\omega_{min}$).

> To generate a USB signal at frequency ω_c:
>
> $$y_{usb}(t) = \sum_{n=1}^{N} A_n \cos(\omega_c + \omega_n)t + \phi_n),$$
>
> - Set the second oscillator frequency to $\omega_s = \omega_c + \omega_o$
> - Set the output summer so that $y_{usb}(t) = f_5(t) + f_6(t)$.

and

> To generate a LSB signal at frequency ω_c:
>
> $$y_{lsb}(t) = \sum_{n=1}^{N} A_n \cos(\omega_c - \omega_n)t - \phi_n),$$
>
> - Set the second oscillator frequency to $\omega_s = \omega_c - \omega_o$
> - Set the output summer so that $y_{lsb}(t) = f_5(t) - f_6(t)$.

3 Weaver Demodulation

While the modulator has a well defined task of translating a well defined low frequency, band-limited signal to a high frequency rf signal, the demodulation process has a particularly "messy" working environment in the sense that its input is potentially the whole rf spectrum, including the AM, FM, TV, etc bands. From the cacophony of signals it must select a narrow-band signal and demodulate it without interference from adjacent (and distant) signals. The Weaver demodulator does an excellent job of this, and is basically a complete SSB communications receiver on its own.

As shown in Fig. 10 the Weaver demodulator is basically the reverse of the steps used in the modulator.

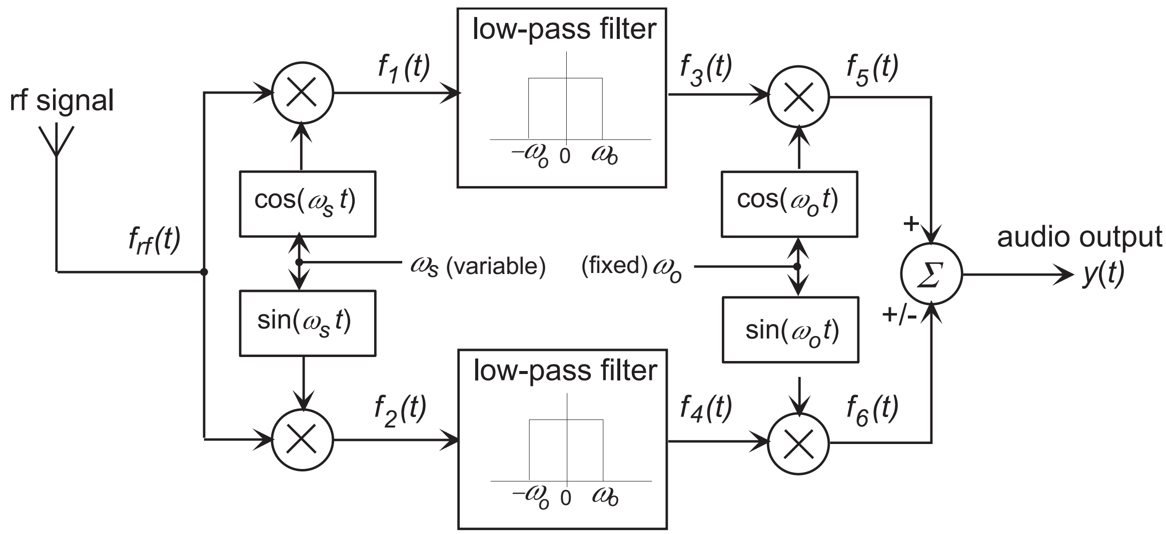

Figure 10: The structure of the Weaver Demodulator.

We start with a "clean" high frequency rf signal $f_{rf}(t) = A_a \cos(\omega_a t + \phi_a)$ .Notice that the major difference from the Weaver modulator is the reversal of the order of the high frequency and low frequency oscillators. The first step is to multiply the input waveform $f_{rf}(t)$ by a pair of variable frequency oscillators ($\cos(\omega_s t$ and $\sin(\omega_s t)$) to translate the input down to "baseband".

1. **At the output of the first multipliers ($\mathbf{f_1(t)}$ and $\mathbf{f_2(t)}$):** Assume that the input is a broadband RF signal with bandwidth ω_{max} as shown in Fig. 11, with a narrowband signal centered on ω_s. The first step is to multiply $f_{rf}(t)$ by the quadrature pair $\cos(\omega_s t)$ and $\sin(\omega_s t)$:

$$\begin{aligned} f_1(t) &= f_{rf}(t) \times \cos(\omega_s t) = A\cos(\omega_a t + \phi_a)\cos(\omega_s t) \\ &= \frac{A}{2}\cos\left((\omega_a - \omega_s)t + \phi_a\right) + \frac{A}{2}\cos\left((\omega_a + \omega_s) + \phi_a\right) \qquad (20) \\ f_2(t) &= f_{rf}(t) \times \sin(\omega_s t) = A\cos(\omega_a t + \phi_a)\sin(\omega_s)t \\ &= -\frac{A}{2}\sin\left((\omega_a - \omega_s)t + \phi_a\right) + \frac{A}{2}\sin\left((\omega_a + \omega_s)t + \phi_a\right) \qquad (21) \end{aligned}$$

The effect of this step is to translate the desired signal down to baseband, as shown in Fig. 11. Note the doubling of the bandwidth in $f_1(t)$ and $f_2(t)$, this will become important when we discuss aliasing in DSP implementations later.

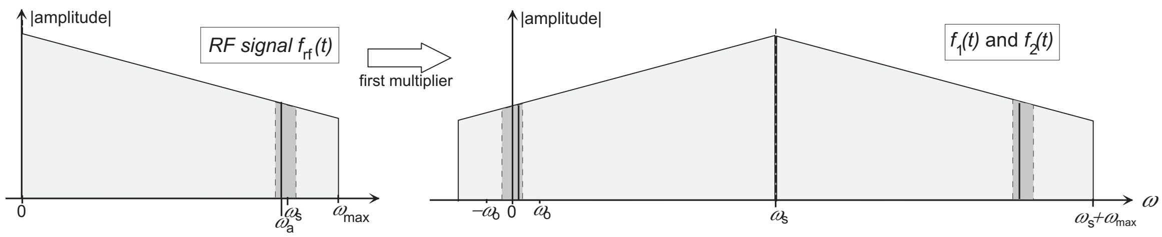

Figure 11: The first step in Weaver demodulation: the broadband rf input signal $f_{rf}(t)$ is multiplied by the quadrature pair $\cos(\omega_s t$ and $\sin(\omega_s t)$. The line represents a component with frequency ω_a in a signal shown as the dark shaded area.

2. Low pass filter outputs: Assume the low-pass filters have a cut-off frequency of ω_o and will pass components in the frequency range $-\omega_o \leq \omega \leq \omega_o$, (see Appendix B). Assume $\omega_s >> \omega_o$ and $\omega_c >> \omega_o$. Clearly the sum terms in Eqs. (20) and (21) and will be rejected by the filter, and the baseband components retained only if they fall within the passband of the filter.

$$f_3(t) = \begin{cases} \dfrac{A}{2}\cos\left((\omega_a - \omega_s)t + \phi_a\right) & \text{if } -\omega_0 \leq \omega_a - \omega_s \leq \omega_o, \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

and

$$f_4(t) = \begin{cases} -\dfrac{A}{2}\sin\left((\omega_a - \omega_s)t + \phi_a\right) & \text{if } -\omega_o \leq \omega_a - \omega_s \leq \omega_o, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

as shown in Fig. 12.

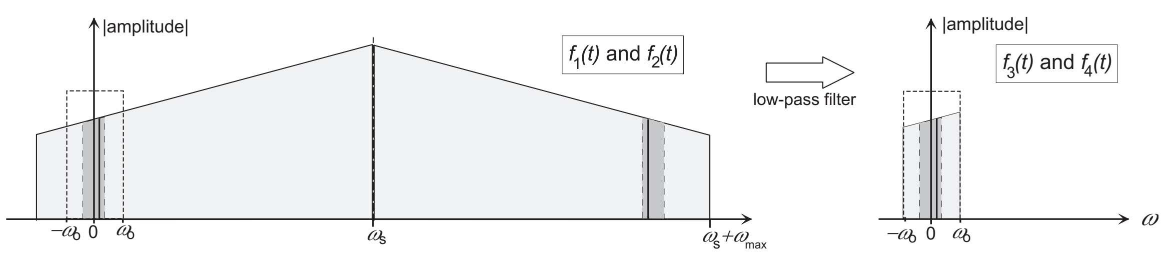

Figure 12: The second step in Weaver demodulation: the action of the low-pass filters to eliminate all components outside the filter passband ($-\omega_o \leq \omega_a \leq \omega_o$).

3. Second multiplier outputs:

$$\begin{aligned} f_5(t) &= f_3(t) \times \cos(\omega_o t) = \frac{A}{2}\cos\left((\omega_a - \omega_s)t + \phi_a\right)\cos(\omega_o t) \\ &= \frac{A}{4}\cos\left(\left(\omega_a - (\omega_s + \omega_o)\right)t + \phi_a\right) + \frac{A}{4}\cos\left(\left(\omega_a - (\omega_s - \omega_o)\right)t + \phi_a\right) \end{aligned} \tag{24}$$

only if $-\omega_o \leq \omega_a - \omega_s \leq \omega_o$, and

$$
\begin{aligned}
f_6(t) &= f_4(t) \times \sin(\omega_o t) = -\frac{A}{2}\sin\left((\omega_a - \omega_s)t + \phi_a\right)\sin(\omega_o t) \\
&= -\frac{A}{4}\cos\left(\left(\omega_a - (\omega_s - \omega_o)\right)t + \phi_a\right) + \frac{A}{4}\cos\left(\left(\omega_a - (\omega_s + \omega_o)\right)t + \phi_a\right) \quad (25)
\end{aligned}
$$

also only if $-\omega_o \leq \omega_a - \omega_s \leq \omega_o$.

4. Summer output: The demodulated audio waveform $y_{audio}(t)$ is found at the output of the summer:

$$
\begin{aligned}
y^{(+)}_{audio}(t) &= f_5(t) + f_6(t) = \frac{A}{2}\cos((\omega_a - (\omega_s - \omega_o))t + \phi_a) \quad (26) \\
y^{(-)}_{audio}(t) &= f_5(t) - f_6(t) = \frac{A}{2}\cos((\omega_a - (\omega_s + \omega_o))t + \phi_a) \quad (27)
\end{aligned}
$$

USB demodulation: If the rf input is a USB signal (Eq. (7))

$$f_{rf}(t) = \sum_{n=1}^{N} A_n \cos((\omega_c + \omega_n)t + \phi_n),$$

with signal frequency ω_c, and we consider just a single component $f_n(t) = A_n \cos((\omega_c + \omega_n)t + \phi_n)$, substitution for ω_a in Eq. (26) gives

$$y^{(+)}_{audio}(t) = \frac{A_n}{2}\cos((\omega_c + \omega_n - (\omega_s - \omega_o))t + \phi_n),$$

and if we select the tuning oscillator frequency $\omega_s = \omega_c + \omega_o$ the output is

$$y^{(+)}_{audio}(t) = \frac{A_n}{2}\cos(\omega_n t + \phi_n)$$

which is the demodulated component, and generalizing to the sum of all such components in the full USB signal:

$$y^{usb}_{demod}(t) = y^{(+)}_{audio}(t) = \frac{1}{2}\sum_{n=1}^{N} A_n \cos(\omega_n t + \phi_n).$$

Summary:
To demodulate a USB signal with frequency ω_c using a low-pass filter with cut-off frequency ω_o,

- Set the tuning oscillator to the frequency $\omega_s = \omega_c + \omega_o$,
- Select the audio output as the sum $y^{(+)}_{audio}(t) = f_5(t) + f_6(t)$.

LSB demodulation: Similarly, if the rf input is a LSB signal (Eq. (8))

$$f(t) = \sum_{n=1}^{N} A_n \cos((\omega_c - \omega_n)t - \phi_n)$$

with signal frequency ω_c, and we consider a single component $f(t) = A_n \cos((\omega_c - \omega_n)t - \phi_n)$, substitution for ω_a in Eq. (27) gives

$$y_{audio}^{(-)}(t) = \frac{A_n}{2} \cos((\omega_c - \omega_n - (\omega_s - \omega_o))t - \phi_n),$$

and when the tuning oscillator frequency $\omega_s = \omega_c - \omega_o$ the output is

$$y_{audio}^{(-)}(t) = \frac{A_n}{2} \cos(\omega_n t - \phi_n)$$

which is the demodulated component. then when applied to all N such components in the full USB signal

$$y_{demod}^{usb}(t) = y_{audio}^{(-)}(t) = \frac{1}{2} \sum_{n=1}^{N} A_n \cos(\omega_n t - \phi_n).$$

Summary:
To demodulate a LSB signal with frequency ω_c using a low-pass filter with cut-off frequency ω_o

- Set the tuning oscillator to the frequency $\omega_s = \omega_c - \omega_o$,
- Select the audio output as the difference $y_{audio}^{(-)}(t) = f_5(t) - f_6(t)$.

4 DSP Implementation Issues

A purely software based implementation of either the modulator or demodulator requires high-speed processing hardware, and is probably not practicable in low cost applications.

4.1 Aliasing Issues

"Aliasing" refers to the inability of DSP systems to represent high frequency signals, and the fact that such high frequency signals will appear as if they have a lower frequency. Any DSP system operates on discrete-time signal samples at intervals of ΔT seconds. Then the sampling frequency is defined as $F_{sample} = 1/\Delta T$ Hz, or $\omega_{sample} = 2\pi F_{sample}$ rad/sec.

Nyquist's *sampling theorem* states that for unambiguous interpretation and processing, *all* digital signals must represent waveforms with no component *at or above* the *Nyquist frequency*, $\omega_N = \omega_{sample}/2$. This includes external signals digitized by an A/D converter, as well as signals created within the processor, such as from digital oscillators and from the outputs from multipliers.

If an attempt is made to digitize or generate a component with a frequency greater than ω_N, it will be "aliased" to a lower frequency below ω_N and will be indistinguishable from any other component at that frequency. Figure 13 shows the apparent frequency of a sinusoid sampled through an A/D converter with a sample rate of ω_{sample}, that is $\omega_N = \omega_{sample}/2$. Below $omega_N$ the apparent frequency is accurate, but in the shaded area where $\omega \geq \omega_N$ the frequency is folded down to an aliased frequency.

The Weaver systems are particularly sensitive to aliasing problems, which impose severe restrictions on the bandwidth of the input and output. These restrictions differ for the modulator and demodulator.

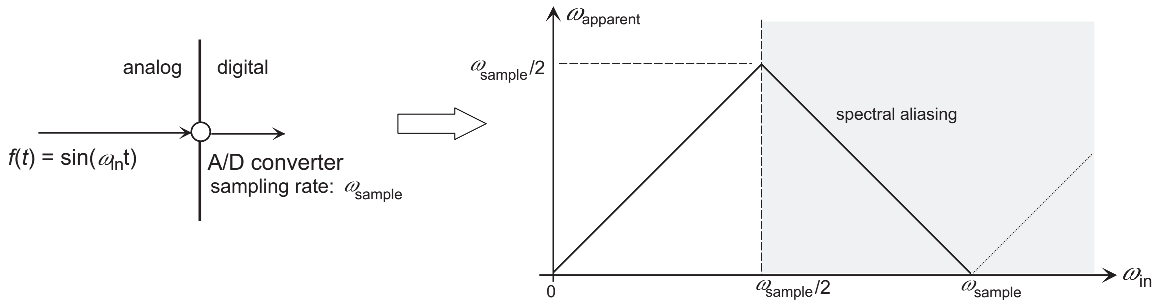

Figure 13: Aliasing at an A/D converter. If the frequency of a sinusoidal input signal, ω_{in}, exceeds the "Nyquist rate", $\omega_{sample}/2$, the apparent frequency is folded down and it appears in subsequent processing as a lower frequency. Note that this is a periodic phenomenon and the above plot repeats every ω_{sample} radians/sec.

4.1.1 Aliasing in the Weaver Modulator

In the modulator the maximum frequency occurs at the outputs of the second multipliers, Eqs. (16) and (17), and are translated directly to the SSB output. These frequencies must satisfy the sampling theorem, with the result is that the Weaver modulator is limited to a signal frequency ω_c

$$\omega_c < \begin{cases} \omega_{sample}/2 & \text{for LSB} \\ \omega_{sample}/2 - 2\omega_o & \text{for USB} \end{cases} \tag{28}$$

where ω_o is the cutoff frequency of the low pass filter. For example a system operating at the CD sampling frequency $F_{sample} = 44.1$ kHz is limited to generating a SSB signal with a highest frequency component below 22.05 kHz. This might typically be a USB signal with a 4 kHz bandwidth at 18 kHz, or a LSB signal with a 4kHz bandwidth at 22kHz.

4.1.2 Aliasing in the Weaver Demodulator

Aliasing in the demodulator poses a slightly more complex problem because of two issues: 1) the input rf signal for the demodulator consists of the entire rf spectrum, and 2) the highest frequencies in the demodulator occur at the output of the first modulator and before the low-pass filters.

(1) The analog rf input must be low-pass filtered to below the Nyquist frequency (known as *pre-aliasing* filtering) before being digitized at the A/D converter. This has the effect of limiting the maximum demodulation bandwidth to below $\omega_{sample}/2$, otherwise the whole rf spectrum will be aliased down into the demodulation bandwidth.

(2) The first pair of oscillators are the high frequency tuning oscillators, ω_s. Their frequency must be limited to below the Nyquist frequency $\omega_s < \omega_{sample/2}$, otherwise spurious components will be generated in the output.

(3) There is an even more severe restriction if *all* aliasing is to be avoided (we will see that this is actually not necessary below). Eqs. (20) and (21), and Fig. 11, show the outputs of the first multipliers have a maximum frequency $\omega_a + \omega_s$, where ω_a is the rf input frequency and ω_s is the first oscillator frequency. But $\omega_s \approx \omega_a$, which implies that aliasing in $f_1(t)$ and $f_2(t)$ will occur if $2\omega_s \geq \omega_{sample}/2$, or if

$$\omega_s \geq \frac{\omega_{sample}}{4}. \tag{29}$$

For example, to avoid *all* aliasing, a DSP demodulator using a sampling frequency of 44.1 kHz must be limited to a rf spectral width of only 11.02 kHz.

It is not necessary, however, to eliminate *all* aliasing. Note that $f_1(t)$ and $f_2(t)$ are the inputs for the two low-pass filters, and therefore the filter outputs, $f_3(t)$ and $f_4(t)$, will be unaffected by aliasing provided *all* aliased components fall above ω_o, the cut-off frequency of the filters and are therefore rejected. Figure 14 shows the situation for a rf spectrum that extends to the Nyquist frequency ω_N, and an oscillator frequency ω_s. The high frequency components are folded down at ω_N and extend to a frequency $\omega_N - \omega_s$. There will be no contamination of the demodulated output provided

$$\omega_s \leq \omega_N - \omega_0, \tag{30}$$

which is a much reduced restriction than that implied by Eq. (29).

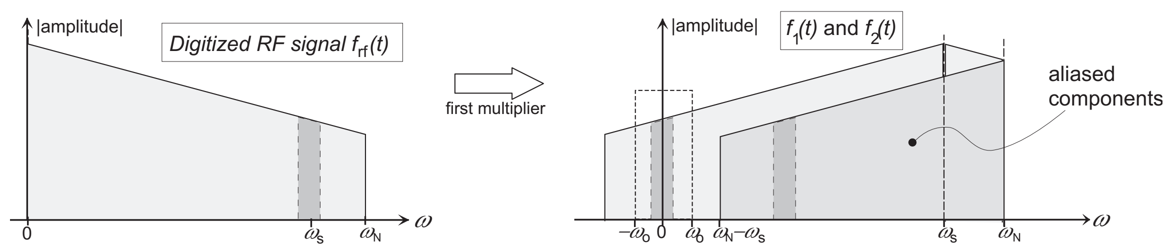

Figure 14: Aliasing in Weaver demodulation using DSP multiplication. This figure should be compared with Fig. 11. Note that the high frequency components (above the Nyquist frequency) have been folded down to a lower frequency and will contaminate the filter outputs if $\omega_s \geq \omega_N - \omega_o$.

For example, for a system sampled at the CD rate (44.1 kHz) and $\omega_o = 2$kHz there will be know aliasing contamination of the output provided the oscillator frequency $\omega_s < 20$ kHz.

4.2 Hybrid Implementations to Overcome Aliasing Problems

4.2.1 Wideband DSP Based SSB Generator

The limiting factor affecting the high frequency use of the Weaver (or any other) DSP modulator is the sampling and processing speed available, and the aliasing generated in the final multiplication stage. There are two approaches that might be taken:

(1) Use a full DSP modulator to generate a SSB signal at as higher frequency as is practicable, then use a simple RF-mixer to up-convert that signal to the RF band. For example, if processing is limited to 192 kHz, generate a SSB signal at $\approx$ 80 kHz, and then convert it to the 40m ham band ($\approx$ 7 MHz).

The problem with this approach that use of a simple mixer will generate an "image" that must be eliminated by RF band-pass filtering.

(2) In Fig. 15 the second multiplier has been replaced by analog hardware that performs the same function, and is thus independent of the available DSP processing speed. The necessary sampling speed for the DSP section is therefore set by the audio bandwidth, $2\omega_o$, and is therefore quite modest.

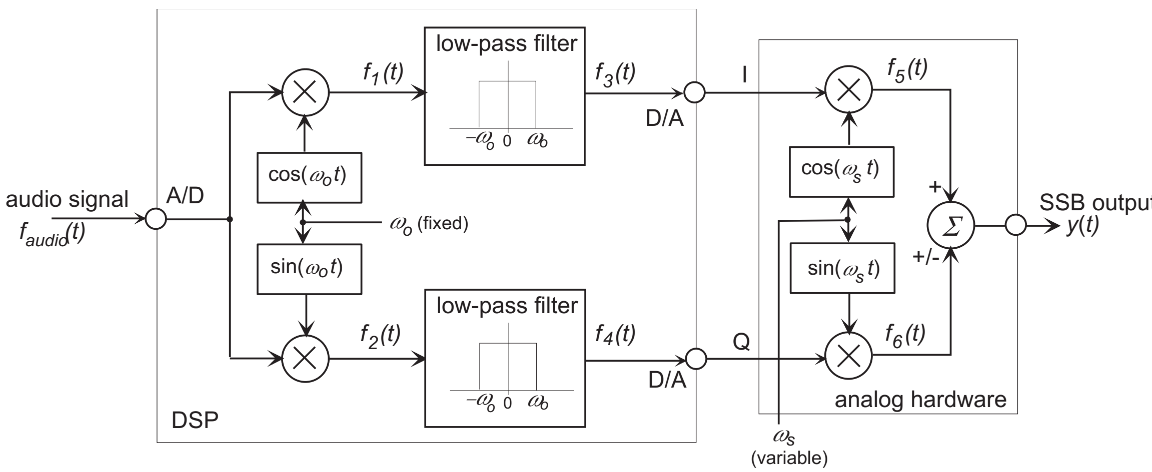

Figure 15: Practical implementation of a Weaver based SSB generator.

4.2.2 Wideband DSP Based SSB Receiver

We saw above that the frequency span of a Weaver based receiver is severely restricted by the sampling rate (and subsequent aliasing) at the A/D converter and first multiplier stage. Almost all high-frequency SDR receivers use a hybrid approach in which the rf signal is down-converted to baseband using analog hardware that is functionally equivalent to the first multiplier stage. The structure is shown in Fig. 16. The pre-alias filters are low-pass, with a cut-off frequency set to the Nyquist frequency of the A/D converters. Note that these filters must be placed before the A/D converters, and cannot be combined with the Weaver filters.

This structure allows for a much reduced DSP processing frequency and complexity. The processing has been reduced to two filters and two multiplications. The highest frequency present will be $2\omega_o$ (in the audio signals), setting a minimum sampling/processing frequency of $4\omega_o$ to prevent aliasing.

4.2.3 A Pseudo-Code Implementation of a Hybrid Weaver Demodulator

Given

a) A pair of identical low-pass filters with a cut-off frequency `Fo` (Hz)

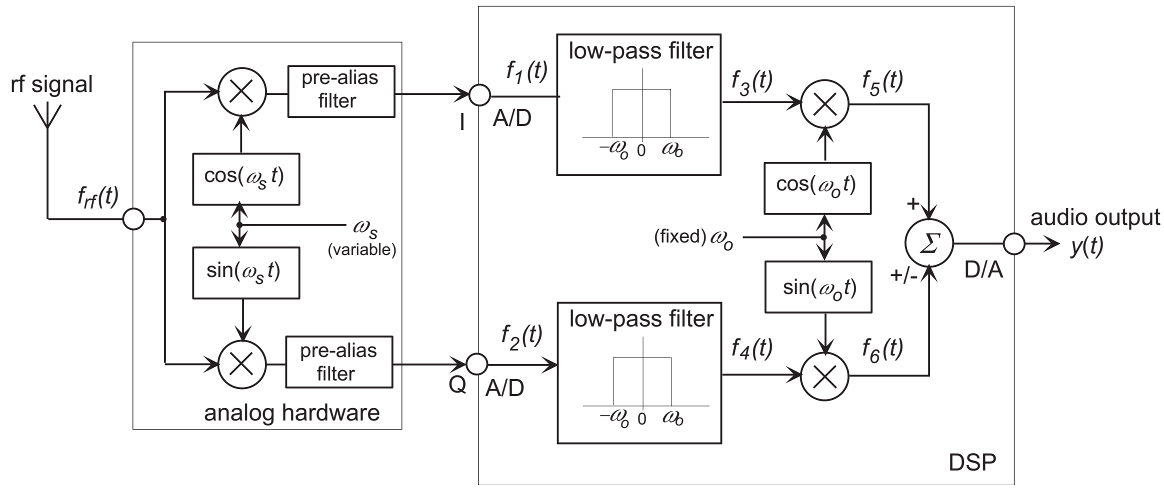

Figure 16: Practical implementation of a Weaver based SSB receiver.

b) A processor sampling rate `Fsample` (Hz)
the following pseudo-code illustrates the essential steps in a Weaver demodulator as shown if Fig. 16:

```
    deltaPhase = 2*pi*Fo/Fsample
    phase = 0
    setClockInterval(1/Fsample)
    waitClock()
loop forever
    f1 = readAD(0)          // Read the two AD converters
    f2 = readAD(1)          // Avoid data skew by reading as close together as possible
    f3 = filterA(f1)        // Low-pass filter
    f4 = filterB(f2)
    f5 = cos(phase) * f3    // Multipliers
    f6 = sin(phase) * f4
    if (USB) then demodOut = f5 + f6     // Summer
    else  demodOut = f5 - f6
    phase = phase + deltaPhase           // Update for next sample
    if phase > 2*pi then phase = phase - 2*pi
    waitClock()                          // Wait for real-time clock
end loop
```

Appendix A: Useful Trigonometric Identities

Products of Sinusoidal Functions:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta) \tag{A.1}$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta) \tag{A.2}$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta) \tag{A.3}$$

$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta) \tag{A.4}$$

These properties are useful for finding the effect of multiplying two time domain sinusoidal functions, for example using Eq(A.1):

$$A_a\cos(\omega_a t + \phi_a) \times \cos(\omega_c t) = \frac{A_a}{2}\left[\cos\left((\omega_a - \omega_c)t + \phi_a\right) + \cos\left((\omega_a + \omega_c)t + \phi\right)\right]$$

which demonstrates the *sum* $(\omega_a + \omega_c)$ and *difference* $(\omega_a - \omega_c)$ frequencies generated by multiplication.

Sums of Angles:

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \tag{A.5}$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{A.6}$$

For example, the *phasing* method for generating a LSB signal $y_{lsb}(t) = A\sin((\omega_s - \omega_a)t)$ is:

$$y_{lsb}(t) = A\sin((\omega_s - \omega_a)t) = A\sin(\omega_c t)\cos(\omega_a t) - A\cos(\omega_c t)\sin(\omega_a t)$$

Negative Angle Formulas:

$$\cos(-\alpha) = \cos(\alpha) \tag{A.7}$$

$$\sin(-\alpha) = -\sin(\alpha) \tag{A.8}$$

These are useful for resolving the issue of apparent occurrences of negative frequencies in a sinusoid. For example if $\omega_a > \omega_b$ in the expression $A\sin((\omega_b - \omega_a)t + \phi)$, the frequency is apparently negative. But by using Eq(A.8), the expression may simply be rewritten as

$$A\sin((\omega_b - \omega_a)t + \phi) = -A\sin((\omega_a - \omega_b)t - \phi)$$

where the frequency as written is now positive.

Angle Shifts by $\pm\frac{\pi}{2}$ ($\pm 90°$) and π ($\pm 180°$):

$$\sin\left(\alpha + \frac{\pi}{2}\right) = \cos(\alpha) \quad , \quad \sin\left(\alpha - \frac{\pi}{2}\right) = -\cos(\alpha) \tag{A.9}$$

$$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin(\alpha) \quad , \quad \cos\left(\alpha - \frac{\pi}{2}\right) = \sin(\alpha) \tag{A.10}$$

$$\sin(\alpha \pm \pi) = -\sin(\alpha) \tag{A.11}$$

$$\cos(\alpha \pm \pi) = -\cos(\alpha) \tag{A.12}$$

These are useful for shifting between representations using sines and cosines.

Appendix B: Signal Processing for Spectral Components with Negative Frequencies

Signal processing operations, such as multiplication, will often generate signals with an apparent negative frequency, for example $f(t) = A\cos(-\omega t + \phi)$. The physical interpretation can be confusing, and some typical questions raised are:

- What the physical meaning of a negative frequency?
- What does a signal with a negative frequency look like on an instrument, such as an oscilloscope or a frequency analyzer?
- What happens if a negative frequency signal is passed through a filter whose frequency response is defined only for positive frequencies?

These are, in fact, non-issues [6]. It is important to remember that a quantity such as $-\omega t + \phi$ is simply a *time-varying angle* inside a trigonometric sin or cos function, and the mathematics of signal processing are completely agnostic to the sign of this angle.

Any confusion may be resolved through the negative angle trigonometric formulas (Eqs. (A.7) and (A.8)) in Appendix A:

$$\cos(-a) = \cos(a) \quad \text{and} \quad \sin(-a) = -\sin(a)$$

which allow any negative frequency component to be written in terms of an equivalent positive frequency, for example

$$f_1(t) = A\cos(-\omega t + \phi) \equiv A\cos(\omega t - \psi) \;\text{ or }\; f_2(t) = A\sin(-\omega t + \psi) \equiv -A\sin(\omega t - \psi).$$

Such substitutions may be made at any point in the analysis, and immediately answer the second question above: a signal with a negative frequency will appear on an instrument as having a positive frequency but with a possible phase/sign change. This is often referred to as "frequency folding", as shown in Fig. 17.

Filtering Signals with Negative Frequency Components: The response of a linear filter (with its frequency response characteristics defined in terms of positive frequencies only) to negative frequency components is an important topic in the discussion of Weaver modulator/demodulators. Both the modulator and demodulator use low-pass filters with a passband defined from $0 - \omega_o$, yet the filter inputs necessarily contain negative frequency components from a prior multiplication. The answer is again found using the negative angle formulas.

For simplicity assume ideal filters, that is the response is either unity in the passband and zero in the stopband. Write the input as a negative frequency sinusoid, and express it as the equivalent positive component:

$$f(t) = A\cos(-\omega t + \phi) \equiv A\cos(\omega t - \phi).$$

[6] The issues do not arise when using complex mathematics where real signal components are described by complex exponentials implicitly containing both positive and negative frequency components.

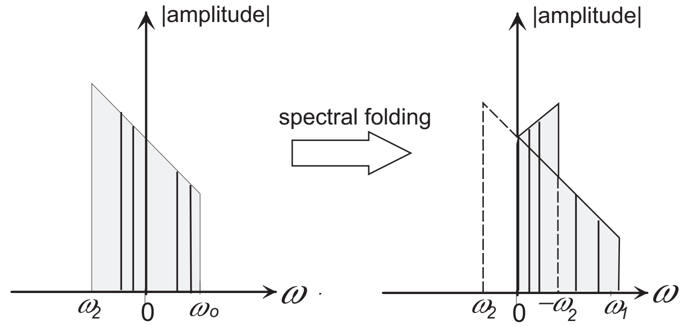

Figure 17: Frequency "folding" of the amplitude spectra about $\omega = 0$ to convert all components to positive frequencies only. Both the original and folded waveform representations are equally valid.

The output $y(t)$ is identical to that for a different input $f'(t) = A\cos(\omega t - \phi)$, that is

$$y(t) = \begin{cases} A\cos(\omega t - \phi) & \text{in the passband} \\ 0 & \text{otherwise} \end{cases}$$

Expressing $y(t)$ back to the original negative frequency input

$$y(t) = \begin{cases} A\cos(-\omega t + \phi) & \text{in } \omega \text{ is in the passband} \\ 0 & \text{otherwise.} \end{cases}$$

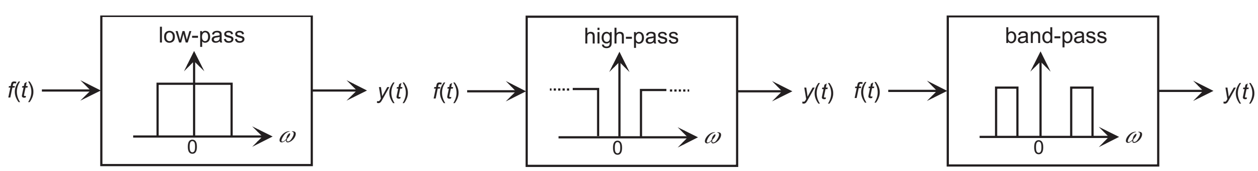

Figure 18: The frequency response of any linear filter is an even function of the input frequency.

The result is that real filters respond symmetrically to positive and negative frequencies as indicated schematically in Fig. 18.